A sequence of thoughts on constructible angles.

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1 Introduction

In classical number theory the algebraic numbers, and their counterparts, the transcendentals, are introduced. An algebraic number is any number that is a root of a polynomial of degree n,

$$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0, \tag{1}$$

with rational coefficients $c_0, c_1, \ldots, c_n$. One proper subset of the algebraic numbers is the set of constructible numbers H. These constructible numbers are those numbers that can be shown to be a length of a segment using only a straight-edge and compass. $1, 2, \sqrt{3}, \frac{22}{7}, \sqrt{\frac{5+\sqrt{5}}{8}}, \ldots$ etc. are all constructible numbers. Of interest in this article is the construction of angles with only a straight-edge and compass. An angle Θ will be called constructible if lengths of leg x and leg y equivalent to the $\cos\Theta$ and $\sin\Theta$ can be constructed. For example $\sqrt{2}$ is constructible by drawing 2 segments of length 1 perpendicular to each other. [1] Connect each of the segments and, by Pythagoras, you have a new segment of length $\sqrt{2}$. Thus an angle of 45^o is constructible. Angles of 60^o and 30^o are also constructible by placing a right triangle of base 1 inside a circle of radius 2 as in Figure 1. $\angle ABC$ is then a 30^o angle and $\angle BAC$ is then a 60^o angle.

2 Content

There are angles that are not constructible with straight-edge and compass. To highlight one, and as a result, hundreds of, nonconstructible integer angle(s) we will pick on an angle of 20^o.

Theorem 2.1 *(The Constructible Roots Theorem for Cubics). If a cubic equation with rational coefficients has no rational root, then it has no constructible root.*

[1] constructing a perpendicular is an accepted construction on any line. place two equidistant points on either side of the point of the desired perpendicular, swing equal, "arbitrary" circles with center at each outside point and connect their two intersections.

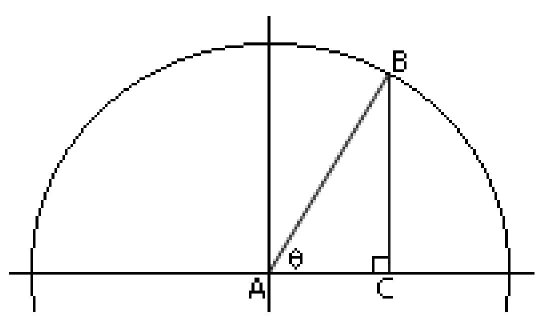

Figure 1: *a 30-60-90 triangle inside a radius 2 circle.*

To show that a given angle is not constructible it is necessary to show that one of the legs required to connect the hypotenuse of the angle are not of constructible lengths. I.e. we will show that an angle of 20^o is not constructible as $\cos 20^o \notin H$. We will create a trigonometric identity for $\cos 3\alpha$ in terms of $\cos\alpha$.

$$\begin{aligned}
\cos 3\alpha &= \cos(2\alpha+\alpha) = \cos 2\alpha\cos\alpha - \sin 2\alpha \sin\alpha && (2)\\
&= (\cos^2\alpha - \sin^2\alpha)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha && (3)\\
&= \cos^3\alpha - \cos\alpha\sin^2\alpha - 2\sin^2\alpha\cos\alpha && (4)\\
&= \cos^3\alpha - 3\cos\alpha sin^2\alpha && (5)\\
&= \cos^3\alpha - 3\cos(1-\cos^2\alpha) && (6)\\
&= 4\cos^3\alpha - 3\cos\alpha && (7)
\end{aligned}$$

Thus when $\alpha = 20^o$ we have

$$\cos 60^o = 4\cos^3 20^o - 3\cos 20^o \qquad (8)$$

let $x = \cos 20^o$ and then, since $\cos 60^o = \frac{1}{2}$ we have

$$\begin{aligned}
\frac{1}{2} &= 4x^3 - 3x \; or && (9)\\
8x^3 - 6x - 1 &= 0 && (10)
\end{aligned}$$

By our algebra, $\cos 20^o$ is a root to this polynomial with rational coefficients and by the Rational Roots Theorem, $\cos 20^o$ must be $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4},\ or\ \pm\frac{1}{8}$. We see $\cos 20^o$ is not any of these fractions as $\cos 0^o > \cos 20^o > \cos 30^o$, and $1 > \cos 20^o > \frac{1}{2}$, so we are left with the fact that $\cos 20^o$ is irrational. We further see that none of $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4},\ or\ \pm\frac{1}{8}$ can be a root of 10 and so, by 2.1, $\cos 20^o \notin H$ and 20^o is not a constructible angle with straight-edge and compass.

The obvious next question is, what angles are constructible? Given an already constructed angle, there are two constructions that you may perform on them to obtain newly constructed angles. They are angle bisection and angle integer multiplication. Without loss of generality we will limit our further discussion to acute angles as obtuse angles can be broken down into acute parts and then treated the same as follows. Bisecting a given angle is done by swinging a circle of radius equal to the minimum length of the legs associated with the angle, drawing a segment between the intersection points on each leg with the circle, and connecting the vertex with the midpoint of that new segment. See Figure 2. For angle integer multiplication, take a given acute angle $\angle BAC$ as in Figure 3.

1. Draw a perpendicular on $\overleftrightarrow{AC}$ at point C, where $|\overline{AC}| = 1$
2. If necessary extend $\overleftrightarrow{AB}$ so that $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{AB}$.
3. At that point make a perpendicular to $\overleftrightarrow{AB}$ and extend it to point D where $|\overline{BD}| = |\overline{AB}| \cdot |\overline{CB}|$.
4. Connect D and A and then since $\triangle BAC \sim \triangle DAB$ we have that $\angle DAC =$ twice $\angle BAC$.

If we want $10 \cdot 30^o$ we add 30^o to itself 10 times and we are done. Thus any angle can be multiplied by any $n \in Z$. Also any angle that is constructible can be halved any number of times, so if we have a 45^o angle we can construct angle $\frac{45}{2^7} = 0.3515625^o$.

Theorem 2.2 *If $n\Theta$, $n \in N$, is not constructible then Θ is not constructible.*

Proof of 2.2 *Let $n\Theta$ be an angle that is not constructible where $n \in N$ and assume Θ is a constructible angle. Then by angle integer multiplication, since Θ is constructible then $h\Theta$ is constructible for all $h \in Z$ and hence $n\Theta$ is constructible. This is a contradiction to our assumption.* □

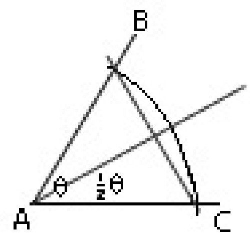

Figure 2: *Angle Bisection.*

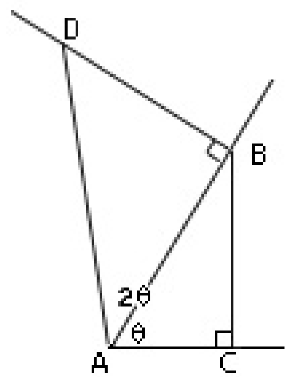

Figure 3: *Acute Angle Integer Multiplication.*

What this theorem gives us is that every integer factor of an integer non constructible angle is also not constructible. We have shown that a 20^o angle is not constructible, so then we have that 10^o, 5^o,4^o,2^o, and 1^o are also not constructible.

Theorem 2.3 *An integer angle is constructible iff it is a multiple of three.*

In order to show this claim, we must first produce a 3^o angle. Thus we introduce a 72^o angle and show that it is, in fact, made of constructible legs.

From trigonometric identities we can find a polynomial equal to $\sin 5\alpha$ in terms of $\sin\alpha$. We see, from algebra and identities that,

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{11}$$

and

$$\cos 3\alpha = (1 - 4\sin^2\alpha)\cos\alpha \tag{12}$$

and

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \tag{13}$$

and

$$\cos 2\alpha = 1 - 2\sin^2\alpha. \tag{14}$$

Thus,

$$\sin 5\alpha = \sin(3\alpha + 2\alpha) = \sin 3\alpha \cos 2\alpha + \cos 3\alpha \sin 2\alpha \tag{15}$$
$$= (3\sin\alpha - 4\sin^3\alpha)(1 - 2\sin^2\alpha) + (1 - 4\sin^2\alpha)\cos\alpha \cdot 2\sin\alpha\cos\alpha \tag{16}$$
$$= 3\sin\alpha - 4\sin^3\alpha - 6\sin^3\alpha + 8\sin^5\alpha + (1 - 4\sin^2\alpha)2\sin\alpha\cos^2\alpha \tag{17}$$
$$= 3\sin\alpha - 10\sin^3\alpha + 8\sin^5\alpha + (2\sin\alpha - 8\sin^3\alpha)(1 - \sin^2\alpha) \tag{18}$$
$$= 3\sin\alpha - 10\sin^3\alpha + 8\sin^5\alpha + 2\sin\alpha - 8\sin^3\alpha - 2\sin^3\alpha + 8\sin^5\alpha \tag{19}$$
$$= 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha. \tag{20}$$

Thus when $\alpha = 72^o$ we have

$$\sin 360^o = 5\sin 72^o - 20\sin^3 72^o + 16\sin^5 72^o. \tag{21}$$

let $x = \sin 72^o$ and then, since $\sin 360^o = 0$ we have after dividing both sides by x,

$$0 = 16x^4 - 20x^2 + 5 \tag{22}$$

We can solve this equation by multiplying by 4 and factoring the right side:

$$5 = 64x^4 - 80x^2 + 25 \tag{23}$$
$$5 = (8x^2 - 5)^2 \tag{24}$$
$$\sqrt{5} = 8x^2 - 5 \tag{25}$$
$$\sqrt{\frac{5+\sqrt{5}}{8}} = x = \sin 72^o. \tag{26}$$

If we draw a tringle with hypotenuse 1 and leg $\sqrt{\frac{5+\sqrt{5}}{8}}$, we then have, by Pythagoras, that

$$1^2 = (\sqrt{\frac{5+\sqrt{5}}{8}})^2 + a^2 \tag{27}$$
$$1 = \frac{10+2\sqrt{5}}{16} + a^2 \tag{28}$$
$$\frac{16}{16} - \frac{10+2\sqrt{5}}{16} = a^2 \tag{29}$$
$$\frac{-2\sqrt{5}+6}{16} = a^2 \ and \tag{30}$$
$$\frac{-1+\sqrt{5}}{4} = a = \cos 72^o. \tag{31}$$

We then have constructible lengths for a 72^o angle. To obtain a 3^o angle, we take a given 30^o angle, bisect it, take the compliment (75^o) and construct our 72^o angle inside that angle. We then take the difference between those two angle and we have a 3^o angle. We now return to finish our proof.

Proof of 2.3 ($\Leftarrow$) *Assume that Θ is an integer multiple of three. Then since a 3^o angle is constructible, by angle integer multiplication, we can construct angle Θ.*
($\Rightarrow$) We can partition all the integers between 1 and 90 into three equivalence classes with the integers mod 3. Then any angle is equal to $3x$, $3x-1$, or $3x-2$ for some $x \in Z$. Assume Θ is constructible and not equal to $3x$ for any $x \in Z$. Then Θ is equal to $3x-1$ or $3x-2$, however we know we can construct angle $3x$ for the same x that is the multiple in Θ. Thus we can construct angle $3x-(3x-1)$ or $3x-(3x-2)$, with angle difference and hence either a 1^o or a 2^o angle is constructible. This is a contradiction so we have that every constructible integer angle is a multiple of three. □

3 Conclusion

Our work up to this point allows us to tell readily that a given angle is constructible or not, and as such that a hexagon (interior 60^o angles) is constructible with a straight-edge and compass, while a nonagon is not since a 40^o angle is not a multiple of three. Thus the following angles are constructible, $\{3, 6, 9, 12, 54, 45, 60, 75, 87, 90, 7.5, .75, 150, .9375, 183, \ldots\}$, generally $3 \cdot \frac{m}{2^n}$ where $\{m, n\} \subseteq Z$, while $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 20, 40, 80, 100, \ldots\}$ are not. Lastly, this more than answers the classic problem of whether there exists a construction that will trisect a given angle, as a 3^o angle clearly is not trisectible, though we see that every angle Θ where $\Theta = 9 \cdot \frac{m}{2^n}$, $\{m, n\} \subseteq Z$, is. Thus 54^o, 18^o,63^o, 117^o, and 2.8125^o are trisectible angles while 51^o,60^o, and 102^o are not.